

Analytic Properties of the Conformal Dirac Operator on the Sphere

Brett Pansano
Northwest Arkansas Community College

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Abstract

In this paper the conformal Dirac operator on the sphere is defined to be operating on the space of $L^2(S^n)$ Clifford algebra-valued functions. The spinorial Laplacian of order d is defined and used to establish Sobolev embedding theorems, and an extension of the conformal Dirac operator to the Sobolev space setting. Interpolation of polynomials in Clifford analysis is left for further investigation.

1 Preliminaries

A Clifford algebra Cl_n can be generated from the standard orthogonal space \mathbb{R}^n with the negative definite inner product, that is, $x^2 = -\|x\|^2$. Then a basis is given by the set of ordered products $e_A = e_{j_1} \cdots e_{j_k}$, where $j_1 < \cdots < j_k$ and $A = \{e_{j_1}, \dots, e_{j_k}\}$ is a subset of \mathbb{N} . For $A = \emptyset$, $e_A = 1$, which is the identity element in Cl_n . A consequence of the equality $x^2 = -\|x\|^2$ is that $\{e_1, \dots, e_n\}$ satisfies the anti-commutation relationship $e_j e_k + e_k e_j = -2\delta_{kj}$, where $1 \leq j, k \leq n$ and δ_{kj} is the Kronecker delta symbol. The Clifford algebra can be considered as being the exterior algebra $\Lambda\mathbb{R}^n$ where the inner-product is added onto the Clifford product, that is, $xy = x \wedge y - \langle x, y \rangle$, for arbitrary elements $x, y \in \mathbb{R}^n$.

Definition 1 *The Dirac operator in \mathbb{R}^n acts on smooth Cl_n -valued functions and is defined by*

$$D_x f(x) = \sum_{i=1}^n e_i \partial_i f(x),$$

where ∂_i is the i -th partial derivative.

Remark 2 *An important point is that the $D_x^2 = -\nabla_x$, where ∇_x is the Laplacian. This follows from the fact that $x^2 = -\|x\|^2$ for $x \in \mathbb{R}^n$. We say that a function f is left (resp. right) monogenic if $Df = 0$ (resp. $fD = 0$).*

We now give several examples of theorems which generalize from classical complex analysis to Clifford analysis. To this end, consider a piecewise- C^1 surface in \mathbb{R}^n , that is, a codimension-1 manifold embedded in \mathbb{R}^n . Let $U \subset \mathbb{R}^n$ be a domain and suppose V is relatively compact in U , denoted $V \subset\subset U$, such that ∂V is a piecewise- C^1 surface. Then we wish to evaluate an integral of the form:

$$\int g(x) \eta(x) f(x) d\sigma(x),$$

where f and g are Clifford algebra-valued functions, $\eta(x)$ is normal to the ∂V at $x \in \mathbb{R}^n$.

Theorem 3 (*Cauchy's Theorem*) Suppose $U \subset \mathbb{R}^n$ is a domain and V is relatively compact with ∂V a piecewise- C^1 hypersurface. Suppose $f, g : U \rightarrow Cl_n$ are right and left monogenic functions, respectively. Then

$$\int g(x) \eta(x) f(x) d\sigma(x) = 0.$$

The function $G : \mathbb{R}^n / \{0\} \rightarrow \mathbb{R}^n / \{0\}$ defined by $G(x) = -\frac{x}{\|x\|^n}$ is the Cauchy kernel in Clifford analysis. The function is infinitely often continuously differentiable and it is homogeneous of degree $1-n$ with respect to the origin. For more details, see for example, [5,12].

Theorem 4 (*Cauchy's Integral Formula*) Let $U \subset \mathbb{R}^n$ be a domain and let V be open and relatively compact in U such that ∂V is a C^1 hypersurface. Suppose f is a left monogenic function on U . Then for $y \in V$,

$$f(y) = \frac{1}{\omega_n} \int_{\partial V} G(x-y) \eta(x) f(x) d\sigma(x),$$

where ω_n is the surface area of the unit sphere S^{n-1} in \mathbb{R}^n .

Next we need to derive the Dirac operator on \mathbb{R}^{n+1} in polar form. For this we follow the approach in [5,12]. Consider polar coordinates, $(r, \omega) \in \mathbb{R}_+ \times S^n$ in \mathbb{R}^{n+1} , where $r = |x| = (x_1^2 + \dots + x_{n+1}^2)^{1/2}$, $\omega = \frac{x}{|x|}$ (a unit vector on S^n). Then the Dirac operator admits the polar decomposition

$$D_x = \frac{\omega}{r} (\Gamma_\omega + r \partial_r).$$

We write this as

$$D_x = \omega \left(\partial_r + \frac{1}{r} \Gamma_\omega \right).$$

Next we define the spinor connection on the sphere.

Definition 5 The spinor connection on the sphere in the x -direction is given $\nabla_x = \partial_x + \frac{1}{2}x\omega$ where $\omega \in S^n$.

In [4] it is shown via the spinor connection it is possible to derive the conformal Dirac operator on the sphere. We state it here as a definition.

Definition 6 *The conformal Dirac operator acting on smooth Clifford algebra-valued functions on the sphere is given by*

$$D_s = \omega \left(\Gamma_\omega - \frac{n}{2} \right),$$

where Γ_ω is the Dirac-Beltrami operator or the gamma operator. See [4,11]

The Dirac-Beltrami operator has the following properties.

Lemma 7 *Let $\omega \in S^n$. Then we have the following intertwining operator identity for the Dirac-Beltrami operator: $\Gamma_\omega \omega + \omega \Gamma_\omega = \omega n$.*

Proof. Using that $D_x^2 = -\nabla_n$, where ∇_n is the Laplacian in \mathbb{R}^{n+1} given by

$$-\nabla_n = -\partial_r^2 - \frac{n}{r} \partial_r - \frac{1}{r^2} \nabla_\omega,$$

∇_ω being up to sign the Laplace-Beltrami operator on S^n . Using this and the commutation relations $[\partial_r, \omega] = 0$ and $[\Gamma_\omega, \partial_r] = 0$, it follows that

$$\begin{aligned} D_x^2 &= \omega \left(\partial_r + \frac{1}{r} \Gamma_\omega \right) \omega \left(\partial_r + \frac{1}{r} \Gamma_\omega \right) \\ &= -\partial_r^2 + \left(\frac{\omega \Gamma_\omega \omega - \Gamma_\omega}{r} \right) \partial_r + \left(\frac{\omega \Gamma_\omega \omega \Gamma_\omega + \Gamma_\omega}{r} \right). \end{aligned}$$

Therefore,

$$\omega \Gamma_\omega \omega - \Gamma_\omega = -n (Id)$$

so that

$$\Gamma_\omega \omega + \omega \Gamma_\omega = \omega n (Id),$$

or simply

$$\Gamma_\omega \omega + \omega \Gamma_\omega = \omega n.$$

■

Remark 8 *In [3] and [4] it is shown that D_s can act on smooth Clifford algebra-valued functions or smooth spinor sections. We are considering that D_s acts on smooth Clifford algebra-valued functions instead of smooth spinor sections. For more details, see, [3].*

2 Spherical Harmonics and Orthogonal Projection on the Sphere

Definition 9 Let \mathbb{H}_m denote the restriction to S^n of the space of Cl_{n+1} -valued harmonic polynomials homogeneous of degree $m \in \mathbb{N} \cup \{0\}$. This is the space of spherical harmonic polynomials homogenous of degree m . Further, let P_m denote the restriction to S^n of left monogenic polynomials homogeneous of degree $m \in \mathbb{N} \cup \{0\}$, and let Q_m denote the restriction to S^n of the space of left monogenic functions of degree $-n - m$ where $m=0,1,2,\dots$.

The set of left monogenic and right monogenic polynomials provided with the obvious laws of pointwise addition and (right) multiplication by Clifford numbers are right Clifford modules.

Definition 10 Let $\mathbb{P}_a(E)$ denote the space of Cl_{n+1} -valued harmonic polynomials of degree $\leq a$, where $a \in \mathbb{N} \cup \{0\}$ on S^n . That is, the space of the restrictions to S^n of all homogeneous harmonic polynomials of degree $\leq a$, where $a \in \mathbb{N} \cup \{0\}$ on \mathbb{R}^{n+1} .

Lemma 11 The space \mathbb{H}_m of Cl_{n+1} -valued spherical harmonic polynomials homogeneous of degree $m \in \mathbb{N} \cup \{0\}$ on \mathbb{R}^{n+1} has dimension

$$N(n, m) = \dim H_m = \dim (Cl_{n+1}) (\dim P_m + \dim Q_m),$$

where

$$\dim P_m = 2^{\lfloor \frac{n+1}{2} \rfloor} \frac{(m+n-1)!}{m!(n-1)!}$$

and

$$\dim Q_m = 2^{\lfloor \frac{n+1}{2} \rfloor} \frac{(m+n-1)!}{m!(n-1)!}$$

Proof. See [5]. ■

To make further progress note

$$N(n, 0) = 1 \text{ and } N(n, m) = 2^{\lfloor \frac{n+1}{2} \rfloor + 1} \frac{(2m+n-1)(m+n-2)!}{(n-1)!m!}$$

and denote by

$$\left\{ Y_{mk}^{(n)} \mid k = 1, \dots, N(n, m) \right\} \quad (1)$$

a fixed $L^2(S^n)$ orthonormal system of Cl_{n+1} -valued spherical harmonic polynomial homogeneous of degree m for $\mathbb{H}_m(S^n)$. In [9] it is shown that Eqn. (1) is an $L^2(S^n)$ orthonormal basis for $\mathbb{H}_m(S^n)$. Moreover, it is shown that

$$L^2(S^n) = \oplus_{m=0}^{\infty} \mathbb{H}_m(S^n) = \sum_{m=0}^{\infty} (P_m \oplus Q_m)(S^n) = \cup_{a=0}^{\infty} \mathbb{H}_a(S^n).$$

Consequently, we have

$$\mathbb{P}_a(S^n) = \oplus_{m=0}^a \mathbb{H}_m(S^n) = \sum_{m=0}^a (P_m \oplus Q_m)(S^n)$$

with

$$\begin{aligned} d_{a,n} &: = \dim(\mathbb{P}_a(E)) = 2^{\lfloor \frac{n+1}{2} \rfloor + 1} \sum_{m=0}^a N(n, m) = 2^{\lfloor \frac{n+1}{2} \rfloor + 1} N(n+1, a) \\ &= 2^{\lfloor \frac{n+1}{2} \rfloor + 1} \frac{(2a+n)(a+n-1)!}{n!a!} \end{aligned}$$

In [1] it is shown that any two spherical harmonics of different degrees are orthogonal, and hence the union of the sets (2) over all $m \in \mathbb{N}_0$ is a complete orthonormal system in the Hilbert module $L^2(S^n)$. Therefore, a function $f \in L^2(S^n)$ can be represented in the $L^2(S^n)$ sense by a generalized Fourier series expansion with respect to this complete orthonormal system of Cl_{n+1} -valued spherical harmonic polynomial homogeneous of degree m :

$$f = \sum_{m=0}^{\infty} \sum_{k=1}^{N(n,m)} \hat{f}_{mk}^{(n)} Y_{mk}^{(n)}$$

with the generalized Fourier coefficients:

$$\hat{f}_{mk}^{(n)} = \left(f, Y_{mk}^{(n)} \right)_{L^2(S^n)} = \int_{S^n} \overline{f(\omega)} Y_{mk}^{(n)}(\omega) dS_n(\omega),$$

where $dS_n(\omega)$ is Lebesgue surface measure on S^n .

3 Polynomial Basis and Orthogonal Projection on the Unit Sphere

Definition 12 Let \mathbb{H}_m denote the restriction to S^n of the space of Cl_{n+1} -valued harmonic polynomials homogeneous of degree $m \in \mathbb{N} \cup \{0\}$. This is the space of spherical harmonic polynomials homogenous of degree m . Further, let P_m denote the restriction to S^n of left monogenic polynomials homogeneous of degree $m \in \mathbb{N} \cup \{0\}$, and let Q_m denote the restriction to S^n of the space of left monogenic functions of degree $-n-m$ where $m=0,1,2,\dots$

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$$N(n, m) = \dim H_m = \dim(Cl_{n+1})(\dim P_m + \dim Q_m),$$

where

$$\dim P_m = 2^{\lfloor \frac{n+1}{2} \rfloor} \frac{(m+n-1)!}{m!(n-1)!}$$

and

$$\dim Q_m = 2^{\lfloor \frac{n+1}{2} \rfloor} \frac{(m+n-1)!}{m!(n-1)!}$$

Proof. See [5]. ■

To make further progress note

$$N(n, 0) = 1 \text{ and } N(n, m) = 2^{\lfloor \frac{n+1}{2} \rfloor + 1} \frac{(2m+n-1)(m+n-2)!}{(n-1)!m!}$$

and denote by

$$\left\{ Y_{mk}^{(n)} \mid k = 1, \dots, N(n, m) \right\} \quad (2)$$

a fixed $L^2(S^n)$ orthonormal system of Cl_{n+1} -valued spherical harmonic polynomial homogeneous of degree m for $\mathbb{H}_m(S^n)$. In [4] it is shown that Eqn. (1) is an $L^2(S^n)$ orthonormal basis for $\mathbb{H}_m(S^n)$. Moreover, it is shown that

$$L^2(S^n) = \oplus_{m=0}^{\infty} \mathbb{H}_m(S^n) = \sum_{m=0}^{\infty} (P_m \oplus Q_m)(S^n) = \cup_{a=0}^{\infty} \mathbb{H}_a(S^n).$$

Consequently, we have

$$\mathbb{P}_a(S^n) = \oplus_{m=0}^a \mathbb{H}_m(S^n) = \sum_{m=0}^a (P_m \oplus Q_m)(S^n)$$

with

$$\begin{aligned} d_{a,n} &: \dim(\mathbb{P}_a(E)) = 2^{\lfloor \frac{n+1}{2} \rfloor + 1} \sum_{m=0}^a N(n, m) = 2^{\lfloor \frac{n+1}{2} \rfloor + 1} N(n+1, a) \\ &= 2^{\lfloor \frac{n+1}{2} \rfloor + 1} \frac{(2a+n)(a+n-1)!}{n!a!} \end{aligned}$$

In [9] it is shown that any two spherical harmonics of different degrees are orthogonal, and hence the union of the sets (1) over all $m \in \mathbb{N}_0$ is a complete orthonormal system in the Hilbert module $L^2(S^n)$. Therefore, a function $f \in L^2(S^n)$ can be represented in the $L^2(S^n)$ sense by a generalized Fourier series expansion with respect to this complete orthonormal system of Cl_{n+1} -valued spherical harmonic polynomial homogeneous of degree m :

$$f = \sum_{m=0}^{\infty} \sum_{k=1}^{N(n,m)} \hat{f}_{mk}^{(n)} Y_{mk}^{(n)}$$

with the generalized Fourier coefficients:

$$\hat{f}_{mk}^{(n)} = \left(f, Y_{mk}^{(n)} \right)_{L^2(E)} = \int_{S^n} \overline{f(\omega)} Y_{mk}^{(n)}(\omega) dS_n(\omega),$$

where $dS_n(\omega)$ is Lebesgue surface measure on S^n . The surface area of S^n in \mathbb{R}^{n+1} is denoted by $S_n(\omega)$, where

$$S_n = |S_n| = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}.$$

Definition 15 The orthogonal projection operator $T_a : L^2(S^n) \rightarrow \mathbb{P}_a(S^n)$ onto $\mathbb{P}_a(S^n)$ may be represented by

$$T_a f = \sum_{m=0}^a \sum_{k=1}^{N(n,m)} \hat{f}_{mk}^{(n)} Y_{mk}^{(n)} = \sum_{m=0}^a \sum_{k=1}^{N(n,m)} \left(f, Y_{mk}^{(n)} \right)_{L^2(S^n)} Y_{mk}^{(n)}.$$

Another form of T_a can be given via the reproducing kernel of $\mathbb{P}_a(S^n)$.

Definition 16 The reproducing kernel of $\mathbb{P}_a(S^n)$ is the uniquely determined kernel $G_a : S^n \times S^n \rightarrow \mathbb{R}$ with the following properties : (i) $G_a(\omega, \cdot) \in \mathbb{P}_a(S^n)$ for every fixed $\omega \in S^n$, (ii) $G_a(\omega, v)$ for all $\omega, v \in S^n$, and (iii) the following reproducing property

$$(f, G_a(\omega, \cdot))_{L^2(S^n)} = f(\omega)$$

for all $\omega \in S^n$ and $f \in \mathbb{P}_a(S^n)$.

Before stating the reproducing kernel G_a we need the following well-known result.

Proposition 17 The addition theorem for spherical harmonic polynomials homogeneous of degree $m \in \mathbb{N}_0$ is given by

$$\sum_{k=1}^{N(n,m)} Y_{mk}^{(n)}(\omega) \otimes Y_{mk}^{*(n)}(\nu) = \frac{1}{\omega_{n+1}} N(n, m) \frac{C_m^{\frac{n-1}{2}}(-\langle \omega, \nu \rangle)}{C_m^{\frac{n-1}{2}}(1)},$$

where $C_m^{\frac{n-1}{2}}$ is the ultra-spherical or Gegenbauer polynomial of degree m with index $\lambda = \frac{n-1}{2}$, where

$$C_m^\lambda(t) = \frac{(2\lambda)_m}{(\lambda+1)_m} P_m^{(\lambda-\frac{1}{2}, \lambda-\frac{1}{2})}(t), \quad t \in [-1, 1].$$

with $(a)_0 = 1$, $(a)_l = a(a+1) \cdots (a+l-1)$, $a \in \mathbb{R}$, $l \in \mathbb{N}$, and $P_m^{(\lambda-\frac{1}{2}, \lambda-\frac{1}{2})} : [-1, 1] \rightarrow \mathbb{R}$ is the Jacobi polynomial $P_m^{(\alpha, \beta)}$ of degree m with indices $\alpha = \beta = \lambda - \frac{1}{2}$. Note that ω_{n+1} is the surface area of the unit sphere S^n in \mathbb{R}^{n+1} .

See [9, p. 10].

Therefore, we can write the $L^2(S^n)$ -orthogonal projection operator T_a onto $\mathbb{P}_a(S^n)$ with the reproducing kernel G_a of $\mathbb{P}_a(S^n)$ as

$$T_a f(\omega) = \int_{S^n} f(\nu) G_a(\omega, \nu) dS_n(\nu) = (f, G_a(\omega, \cdot))_{L^2(S^n)},$$

which is a form of the Cauchy Integral Formula in Clifford Analysis.

Definition 18 For $d \in \mathbb{N}$, the spinorial Laplacian of order $d > 0$ on S^n is the operator defined by

$$(D_{\mathbf{s}}^2)^d = \left(\Gamma_{\omega} - \frac{n}{2}\right)^{2d}$$

We need the following lemma from Balinski and Ryan [1]. We reproduce the proof here for completeness.

Lemma 19 The spectral resolution of the conformal Dirac operator D_s on the unit sphere S^n is given by

$$\sigma(D_{\mathbf{s}}) = \left\{ \left(m + \frac{n}{2}\right); m = 0, 1, 2, \dots \cup -\left(m + \frac{n}{2}\right); m = 0, 1, 2, \dots \right\}$$

and where the eigenvectors are $p_m \in P_m$ and $q_m \in Q_m$.

Proof. Let $\phi : S^n \rightarrow Cl_{n+1}$ be a C^1 function. Then $\phi \in L^2(S^n)$ and

$$\phi(\omega) = \sum_{m=0}^{\infty} \sum_{j=0}^{\dim P_m} p_{mj}(\omega) + \sum_{m=0}^{-\infty} \sum_{j=0}^{\dim Q_m} q_{mj}(\omega),$$

where $p_{mj} \in P_m$ and $q_{mj} \in Q_m$ are eigenvectors of the Dirac-Beltrami operator Γ_{ω} . Further they may be chosen so that within P_m they are mutually orthogonal. This is similar for the eigenvectors in Q_m . As $\phi \in C^1$ then $D_s \phi \in C^0(S^n)$ and so $D_s \phi \in L^2(S^n)$. Consequently,

$$D_s \phi = \omega \left(\sum_{m=0}^{\infty} \left(m + \frac{n}{2}\right) \sum_{j=0}^{\dim P_m} p_{mj}(\omega) + \sum_{m=0}^{\infty} \left(-m - \frac{n}{2}\right) \sum_{j=0}^{\dim Q_m} q_{mj}(\omega) \right).$$

But $\omega p_m(\omega) \in Q_m$ and $\omega q_m(\omega) \in P_m$. Consequently,

$$D_s \phi = \sum_{m=0}^{\infty} \left(m + \frac{n}{2}\right) \sum_{j=0}^{\dim Q_m} q_{mj}(\omega) + \sum_{m=0}^{\infty} \left(-m - \frac{n}{2}\right) \sum_{j=0}^{\dim P_m} p_{mj}(\omega).$$

Therefore, the spectrum, $\sigma(D_{\mathbf{s}})$, of the conformal Dirac operator D_s on the unit sphere S^n is given by

$$\sigma(D_{\mathbf{s}}) = \left\{ \left(m + \frac{n}{2}\right); m = 0, 1, 2, \dots \cup -\left(m + \frac{n}{2}\right); m = 0, 1, 2, \dots \right\}.$$

■

4 Sobolev Space Estimates in Clifford Analysis on the Unit Sphere

The embedding theorems include various inclusions between the Sobolev spaces L_s^2 .

Definition 20 For any non-negative integer k , let $C^k(S^n)$ denote the space of k -times continuously differentiable functions on S^n equipped with the uniform C^k -norm, defined for $\varphi \in C^k(S^n)$ by

$$\|\varphi(\omega)\|_{C^k(S^n)}^2 = \sum_{j=0}^k \sup_{\omega \in TS^n \setminus \{0\}} |\nabla^j \varphi(\omega)|^2,$$

Note that TS^n is the tangent space at $\omega \in S^n$ and ∇ is the spinor connection. The subspace of compactly supported functions is denoted by $C_0^k(S^n)$.

Definition 21 For a non-negative real number s , we define the Sobolev spaces $L_s^2(S^n)$ as the closure of $\oplus_{m=0}^{\infty} \mathbb{H}(S^n)$ with respect to the s^{th} Sobolev norm

$$\|\varphi\|_{L_s^2(S^n)}^2 = \sum_{m=0}^{\infty} \left(m + \frac{n-1}{2}\right)^{2s} \sum_{k=1}^{N(n,m)} |\hat{\varphi}_{mk}^{(n)}|^2$$

The space L_s^2 is a Hilbert space with the inner product

$$(\varphi, \psi)_{L_s^2(S^n)} = \sum_{m=0}^{\infty} \left(m + \frac{n-1}{2}\right)^{2s} \sum_{k=1}^{N(n,m)} \hat{\varphi}_{m,k}^{(n)} \hat{\psi}_{m,k}^{(n)} \text{ for } \varphi, \psi \in L_s^2(S^n),$$

which induces the norm $\|\cdot\|_{L_s^2(S^n)}$. Thus

$$L_s^2(S^n) = \left\{ \varphi \in S^n : \|\varphi\|_{L_s^2(S^n)}^2 < \infty \right\},$$

where $\varphi = \sum_{m=0}^{\infty} \sum_{k=1}^{N(n,m)} \hat{\varphi}_{mk}^{(n)} Y_{mk}^{(n)}$ with inner product $\langle \varphi, \psi \rangle_{L_s^2(S^n)}$.

Remark 22 For $s > \frac{n}{2}$ there exists a constant c_s such that

$$\|f\|_{C(S^n)} \leq c_s \|f\|_{L_s^2(S^n)}, \quad \forall f \in L_s^2(S^n),$$

that is, $L_s^2(S^n)$ is embedded in $C(S^n)$. Also, for $s > \frac{n}{2}$ the space $L_s^2(S^n)$ is a reproducing kernel Hilbert space. Note that the spaces $L_s^2(S^n)$ are nested, that is, $L_t^2(S^n) \subset L_s^2(S^n)$ whenever $t \geq s$.

Definition 23 We denote by

$$\mathbb{P}_a^\perp(S^n) = \left\{ f \in L^2(S^n) : (f, g)_{L^2(S^n)} = 0 \quad \forall g \in \mathbb{P}_a(S^n) \right\}$$

the orthogonal complement of $\mathbb{P}_a(S^n)$ in $L^2(S^n)$.

The orthogonal complement of $\mathbb{P}_a(S^n)$ in $L_s^2(S^n)$, that is the space of all those functions in $L^2(S^n)$ which are $L^2(S^n)$ -orthogonal to $\mathbb{P}_a(S^n)$. From the definition of the inner product $(\cdot, \cdot)_{L_s^2(S^n)}$, the orthogonal complement of $\mathbb{P}_a(S^n)$ in $L_t^2(S^n)$ is simply $\mathbb{P}_a^\perp(S^n) \cap L_t^2(S^n)$.

The following estimates hold for either sections in $\mathbb{P}_a(S^n)$ or in $\mathbb{P}_a^\perp(S^n) \cap L_s^2(S^n)$.

Lemma 24 *The following estimates hold in the Sobolev spaces $L_s^2(S^n)$.*

1. *Let $s \geq 0$. Let $\varphi \in \mathbb{P}_a(S^n)$. Then $L_0^2(S^n) \subset L_s^2(S^n)$, that is*

$$\|\varphi\|_{L_s^2(S^n)} \leq \left(d + \frac{n-1}{2}\right)^s \|\varphi\|_{L_0^2(S^n)}.$$

2. *Let $s \geq t \geq 0$. Then for any $\varphi \in \mathbb{P}_a^\perp(S^n) \cap L_s^2(S^n)$,*

$$\|\varphi\|_{L_t^2(S^n)} \leq \left(d + \frac{n-1}{2}\right)^{t-s} \|\varphi\|_{L_s^2(S^n)}.$$

3. *Let k be a non-negative integer and let $s - \frac{n}{2} \geq k$. Then $L_s^2(S^n)$ embeds continuously in $C^k(S^n)$, that is,*

$$L_s^2(S^n) \subset C^k(S^n).$$

If, moreover, $s - \frac{n}{2} > k$, then the embedding is also compact.

Proof. For (1), let $\varphi \in \mathbb{P}_a(S^n)$. Then we have

$$\begin{aligned} \|\varphi\|_{L_s^2(S^n)}^2 &= \sum_{m=0}^a \sum_{j=1}^{N(n,m)} \left(m + \frac{n-1}{2}\right)^{2s} |\hat{\varphi}_{mj}^{(n)}|^2 \\ &\leq \left(a + \frac{n-1}{2}\right)^{2s} \|\varphi\|_{L_0^2(S^n)}^2 \end{aligned}$$

pointwise. For (2), for any $\varphi \in \mathbb{P}_a^\perp(S^n) \cap L_s^2(S^n)$ we have for $s \geq t \geq 0$,

$$\begin{aligned} \|\varphi\|_{L_t^2(S^n)}^2 &= \sum_{m=a+1}^{\infty} \sum_{j=1}^{N(n,m)} \left(m + \frac{n-1}{2}\right)^{2t} |\hat{\varphi}_{mj}^{(n)}|^2 \\ &\leq \left(a + 1 + \frac{n-1}{2}\right)^{2(t-s)} \sum_{m=a+1}^{\infty} \sum_{j=1}^{N(n,m)} \left(m + \frac{n-1}{2}\right)^{2s} |\hat{\varphi}_{mj}^{(n)}|^2 \\ &\leq \left(a + 1 + \frac{n-1}{2}\right)^{2(t-s)} \|\varphi\|_{L_s^2(S^n)}^2, \end{aligned}$$

which implies that the embedding is bounded.

To show that the embedding is compact, let I denote the inclusion map from $L_s^2(S^n)$ to $L_t^2(S^n)$. Then for $\varphi \in L_t^2(S^n)$, we have

$$\|(T_a - I)\varphi\|_{L_t^2(S^n)} \leq \left(a + \frac{n-1}{2}\right)^{t-s} \|\varphi\|_{L_s^2(S^n)}$$

since $(T_a - I)\varphi \in \mathbb{P}_a^\perp(S^n) \cap L_s^2(S^n)$. Observe that as $a \rightarrow \infty$, $\|(T_a - I)\varphi\|_{L_t^2(S^n)} \rightarrow 0$. Now since the limit of finite rank operators is a compact operator we see that the embedding of $L_s^2(S^n)$ into $L_t^2(S^n)$ is compact.

For (3), consider first the case $k = 0$. We must estimate the sup-norm of a smooth function on S^n in terms of the $L_s^2(S^n)$ norm. Let $\varphi \in \mathbb{P}_a^\perp(S^n) \cap L_s^2(S^n)$. Then we have

$$\varphi(\omega) = \sum_{m=a+1}^{\infty} \sum_{j=1}^{N(n,m)} \hat{\varphi}_{mj}^{(n)} Y_{mj}^{(n)}(\omega), \quad \omega \in S^n.$$

Assume this Fourier series is uniformly convergent when $s > \frac{n}{2}$, so that it is also true in the pointwise sense. Hence, it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned} |\varphi(\omega)| &= \left| \sum_{m=a+1}^{\infty} \sum_{j=1}^{N(n,m)} \hat{\varphi}_{mj}^{(n)} Y_{mj}^{(n)}(\omega) \right| \\ &\leq \sum_{m=a+1}^{\infty} \sum_{j=1}^{N(n,m)} \left[\left(m + \frac{n-1}{2} \right)^s |\hat{\varphi}_{mj}^{(n)}| \right] \left[\left(m + \frac{n-1}{2} \right)^{-s} |Y_{mj}^{(n)}(\omega)| \right] \\ &\leq \sum_{m=a+1}^{\infty} \sum_{j=1}^{N(n,m)} \left[\left(m + \frac{n-1}{2} \right)^{2s} |\hat{\varphi}_{mj}^{(n)}|^2 \right]^{1/2} \sum_{m=a+1}^{\infty} \sum_{j=1}^{N(n,m)} \left[\left(m + \frac{n-1}{2} \right)^{-2s} |Y_{mj}^{(n)}(\omega)|^2 \right]^{1/2} \\ &\leq \|\varphi\|_{L_s^2(S^n)} \left(\sum_{m=a+1}^{\infty} \sum_{j=1}^{N(n,m)} \left[\left(m + \frac{n-1}{2} \right)^{-2s} |Y_{mj}^{(n)}(\omega)|^2 \right] \right) \\ &= \|\varphi\|_{L_s^2(S^n)} \left(\frac{1}{\omega_n} \sum_{m=a+1}^{\infty} \frac{N(n,m)}{\left(m + \frac{n-1}{2} \right)^{2s}} \right)^{1/2}. \end{aligned}$$

Observe that there exists positive constants c_1, c_2 independent of m such that

$$c_1 \left(m + \frac{n-1}{2} \right)^{n-1} \leq N(n,m) \leq c_2 \left(m + \frac{n-1}{2} \right)^{n-1}.$$

By the integral test, the second term in the sum is finite if and only if $s > n/2$.

Furthermore, the sum satisfies

$$c_3 \left(m + \frac{n-1}{2} \right)^{n-2s} \leq \frac{1}{\omega_n} \sum_{m=a+1}^{\infty} \frac{N(n,m)}{\left(m + \frac{n-1}{2} \right)^{2s}} \leq c_4 \left(m + \frac{n-1}{2} \right)^{n-2s}$$

since

$$c_3 \left(m + \frac{n-1}{2} \right)^{n-2s} \leq \lim_{M \rightarrow \infty} \int_a^M \left(x + \frac{n-1}{2} \right)^{n-2s-1} dx \leq c_4 \left(m + \frac{n-1}{2} \right)^{n-2s}$$

for appropriate constants c_3, c_4 independent of a . Therefore, for $s > n/2$ with the positive constant $c_5 = \sqrt{c_4}$

$$\sup_{\omega \in S^n} |\varphi(\omega)| \leq c_5 \left(a + \frac{n-1}{2} \right)^{n-2s} \|\varphi\|_{L_s^2(S^n)}$$

which is the case for $k = 0$.

For the general case, fix k and choose $\varphi \in \mathbb{P}_a^\perp(S^n) \cap L_s^2(S^n)$ with $s - n/2 \geq k$. For any $\alpha \leq k$ where $\alpha \in \mathbb{N}$, we have

$$\begin{aligned}
\|D_{\mathbf{s}}^\alpha \varphi(\omega)\|_{L_{s-\alpha}^2(S^n)} &= \left\| \sum_{m=a+1}^{\infty} \sum_{j=1}^{N(n,m)} \hat{\varphi}_{mj}^{(n)} \left(D_{\mathbf{s}}^\alpha Y_{mj}^{(n)}(\omega) \right) \right\|_{L_{s-\alpha}^2(S^n)} \\
&\leq \sum_{m=a+1}^{\infty} \sum_{j=1}^{N(n,m)} \left[\left(m + \frac{n-1}{2} \right)^s |\hat{\varphi}_{mj}^{(n)}| \right] \left[\left(m + \frac{n-1}{2} \right)^{-s} \left(m + \frac{n}{2} \right)^\alpha |Y_{mj}^{(n)}(\omega)| \right] \\
&\leq \sum_{m=a+1}^{\infty} \sum_{j=1}^{N(n,m)} \left[\left(m + \frac{n-1}{2} \right)^{2s} |\hat{\varphi}_{mj}^{(n)}|^2 \right]^{1/2} \cdot \left[\left(m + \frac{n-1}{2} \right)^{-2s} \left(m + \frac{n}{2} \right)^{2\alpha} |Y_{mj}^{(n)}(\omega)|^2 \right]^{1/2} \\
&\leq \|\varphi\|_{L_s^2(S^n)} \left(\frac{1}{\omega_n} \sum_{m=a+1}^{\infty} \frac{N(n,m) \left(m + \frac{n}{2} \right)^{2\alpha}}{\left(m + \frac{n-1}{2} \right)^{2s}} \right)^{1/2}.
\end{aligned}$$

Observe that there exists constants \hat{c}_1 and \hat{c}_2 such that

$$\hat{c}_1 \left(m + \frac{n-1}{2} \right)^\alpha \leq \left(m + \frac{n}{2} \right)^\alpha \leq \hat{c}_2 \left(m + \frac{n-1}{2} \right)^\alpha$$

for all $\alpha \in \mathbb{N}$. As before, there exists positive constants \hat{c}_3 and \hat{c}_4 such that \hat{c}_3

$$\hat{c}_3 \left(m + \frac{n-1}{2} \right)^{n-2s+2\alpha} \leq \frac{1}{\omega_n} \sum_{m=a+1}^{\infty} \frac{N(n,m)}{\left(m + \frac{n-1}{2} \right)^{2s-2\alpha}} \leq \hat{c}_4 \left(m + \frac{n-1}{2} \right)^{n-2s-2\alpha}.$$

Therefore, for $\alpha < s - n/2$, we have

$$\|D_{\mathbf{s}}^\alpha \varphi(\omega)\|_{L_{s-\alpha}^2(S^n)} \leq \hat{c}_5 \left(a + \frac{n-1}{2} \right)^{\frac{n}{2}-s+\alpha} \|\varphi\|_{L_s^2(S^n)};$$

with positive constants $\hat{c}_5 = \sqrt{\hat{c}_4}$. This implies that $D_{\mathbf{s}}^\alpha : L_s^2(S^n) \rightarrow L_{s-\alpha}^2(S^n)$ is continuous. By the first part of the proof, $D_{\mathbf{s}}^\alpha \varphi \in C^0(S^n)$ for all $\alpha \leq k$, so that $f \in C^k(S^n)$. ■

The following estimates should be able to be used in polynomial interpolation on the unit sphere in Clifford analysis. For more details, see, [6].

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